High Accuracy Sub-Pixel Image Registration under Noisy Condition

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***Abstract*—Image registration plays an important role in many image processing applications. A key problem is that the accuracy of registration can be severely affected by noise. This paper presents a sub-pixel registration method for noisy images. In particular, we investigate how to estimate transitional shift to a high precision using the noisy phase data in frequency domain. Based on theoretical analysis, we find that the noise-caused phase change for every frequency component of high signal-to-noise ratio (SNR) can be well approximated by a Gaussian distribution. Furthermore, we show that the reliability of phase data can also be measured by the SNR of corresponding frequency component. A noise-robust registration framework is proposed to utilize high-SNR frequency components adaptively, while masking out the components of SNR lower than a threshold. Experiments demonstrate that the proposed method is superior to existing image registration methods in the presence of noise.**

***Keywords*—Noisy image registration; phase difference plane; weighted least squares; distribution of the phase change**

I. INTRODUCTION

Image registration is a fundamental and challenging problem in image processing and is of vital importance in many fields such as video coding, super-resolution and medical imaging. It refers to aligning two images with pixel or sub-pixel accuracy. Registration methods can be roughly divided into two classes: spatial domain methods and frequency domain methods.

The spatial domain methods vary from interpolation-based, feature-based, to Taylor expansion-based and optimization-based methods. One example of the interpolation-based methods is the correlation interpolation [1]. The accuracy of the method depends highly on the quality of the interpolation algorithms and the computational complexity is generally high. Based on the Taylor expansions of image, Keren et al. [2] proposed a method which is often used in super-resolution. However, it can only handle small shifts. In order to overcome this shortcoming, Bergen et al. [3] developed a hierarchical framework to estimate shifts in a multi-resolution structure.

On the other hand, the frequency domain methods have its benefits. Based on the Fourier shift theorem, translation in the spatial domain corresponds to a simple phase shift in the Fourier domain. The computational complexity of the methods is usually low due to the use of FFT. Reddy and Chatterji. [4] propose such shift and rotation estimating method. Murat Balci et al. [5] and Jinchang Ren et al. [6] apply a phase correlation technique to estimate the translational shifts. Vandewalle et al. [7] use the low frequency spectrum to avoid the effects of aliasing when estimating the shifts in frequency domain.

Image processing applications such as super-resolution need high precision when aligning the images. Most of the registration methods mentioned above typically assume that the images are free of noise. Although some methods can cope with low level disturbance, the performances of most image registration methods will become worse in the presence of noise. In other word, most of the methods are not robust with respect to noise. This prompts us to design the following high-precision noise-robust registration algorithm.

For this paper, we use the noise-disturbed phase data to estimate the sub-pixel translational shifts between noisy images. We will discuss how Gaussian noise disturbs the phase of frequency components and find out that frequency components are disturbed in different degrees. In the following, we will show that the registration problem is equal to fitting a phase plane using a weighted least squares estimator and different frequency components make different contributions in the computation. The paper points out that the noise will bring about extra phase change and we call it Noise-induced phase change here. By analyzing the distribution of the Noise-induced phase change, we show that the probability density function of the Noise-induced phase change is approximately Gaussian under certain conditions and the variance is proportional to the noise to signal ratio. So the weights can be easily computed by the noise to signal ratio. Unlike other estimating algorithms whose performance is often very low for noise, our methods outperform the existing registration methods, especially under the circumstance of heavy noise.

This paper is organized as follows. In the next section, we describe necessary theory about aligning clean images. Section III presents our extension of Section II to cope with noisy images. Experimental results are then given in Section IV and Section V ends the paper with a few remarks.

II. IMAGE REGISTRATION WITHOUT NOISE

The motivation of our method relies on the Fourier shift theorem. In this section, we will explain how to cope with clean image registration based on the theorem.

Assume we have two images $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ with horizontal and vertical shifts, Δx_1 and Δx_2 . The relationship between the image pair in spatial domain can be described as follows:

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$$f_2(\mathbf{x}) = f_1(\mathbf{x} + \Delta\mathbf{x}) \quad (1)$$

with $\mathbf{x} = [x_1 \quad x_2]^T$, $\Delta\mathbf{x} = [\Delta x_1 \quad \Delta x_2]^T$.

Their corresponding Fourier transforms F_1 and F_2 will be related by a linear phase shift:

$$F_2(\mathbf{u}) = e^{j2\pi\mathbf{u}^T\Delta\mathbf{x}} F_1(\mathbf{u}) \quad (2)$$

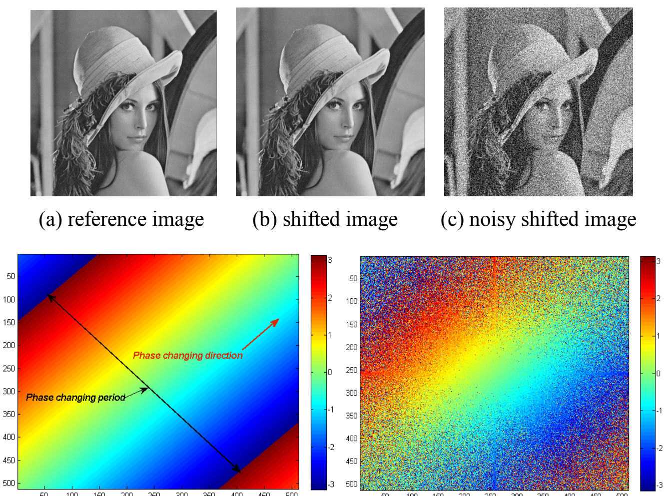
where $\mathbf{u}$ is the frequency value. So we have the following equations:

$$|F_2(\mathbf{u})| = |F_1(\mathbf{u})| \quad (3)$$

$$2\pi\mathbf{u}^T\Delta\mathbf{x} = \angle(F_2(\mathbf{u})/F_1(\mathbf{u})) \quad (\bmod 2\pi) \quad (4)$$

As we all know, the shift parameter vector $\Delta\mathbf{x}$ can be ideally estimated via a few frequency components (at least three) based on equation (4).

In fact, equation (4) denotes a plane with slope $\Delta\mathbf{x}$ and we call it phase difference plane here. As illustrated by Fig. 1(d), the red line and the black line represent the changing direction and changing period of the plane, respectively. The changing direction is determined by the angle of vector $\Delta\mathbf{x}$ while the changing period is associated with the norm of vector $\Delta\mathbf{x}$.



(a) reference image (b) shifted image (c) noisy shifted image

(d) phase difference plane (e) noisy phase difference plane

Fig. 1. Phase difference plane of image pair

In practice, image collection may have small errors. So for the purpose of making the solution less sensitive, we generally use a least squares estimator to obtain the slope of the phase difference plane. Then we have the objective function as follows:

$$\Delta\mathbf{x} = \arg\min_{\Delta\mathbf{x}} \|2\pi \cdot \mathbf{U} \cdot \Delta\mathbf{x} - \mathbf{\Psi}\|_2^2 \quad (5)$$

$\mathbf{U}$ and $\mathbf{\Psi}$ are as follows where i denotes the i^{th} frequency point.

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_i^T \\ \vdots \\ \mathbf{u}_N^T \end{pmatrix}, \quad \mathbf{\Psi} = \begin{pmatrix} \angle(F_2(\mathbf{u}_1)/F_1(\mathbf{u}_1)) \\ \vdots \\ \angle(F_2(\mathbf{u}_i)/F_1(\mathbf{u}_i)) \\ \vdots \\ \angle(F_2(\mathbf{u}_N)/F_1(\mathbf{u}_N)) \end{pmatrix}$$

III. Proposed Noise-Robust Image Registration Method

In this section, we present our extension of the scheme in Section II to noisy image registration.

Here we reuse the motivation described above. In Section II, a least square method is generally used to fit the phase difference plane. This maybe works when the image is clean or a bit noisy. If the noise level increases, the registration problem turns to another case.

Assume $F(\mathbf{u}) = |F(\mathbf{u})| \cdot e^{j\varphi(\mathbf{u})}$ are frequency components of one image. Under the influence of Gaussian noise, these components will be disturbed. Consequently, the phase difference plane becomes noisy as shown in Fig. 1(e). We find out that the low-frequency components in the center are cleaner than high-frequency ones. It may be because the amplitude of low-frequency component is generally much larger and can't be easily changed by noise. This illustrates that we shouldn't treat all the components equally as (5). In order to fit the phase difference plane and obtain the slope $\Delta\mathbf{x}$, those slightly disturbed frequency components should contribute more while those severely disturbed ones contribute less.

Based on above intuitive analysis, we'd better use a weighted least square method instead. To support the intuition, we have the following theoretical analysis.

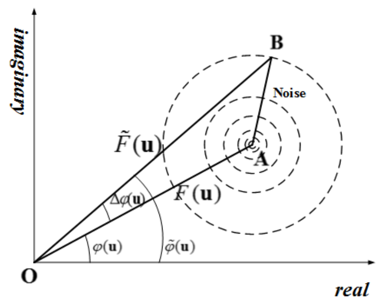


Fig. 2. Phase error caused by noise

As illustrated by Fig. 2, $F(\mathbf{u})$ is disturbed by Gaussian noise (vector $\overrightarrow{AB}$) and becomes $\tilde{F}(\mathbf{u})$. The Noise-induced phase change $\Delta\varphi(\mathbf{u})$ is:

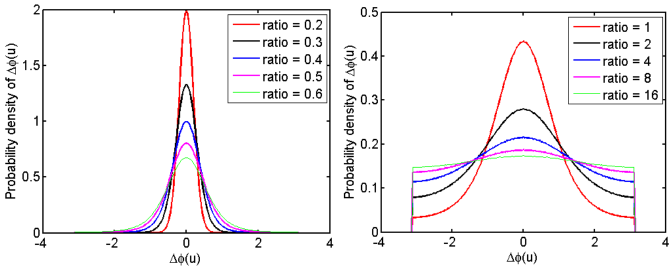
$$\Delta\varphi(\mathbf{u}) = \tilde{\varphi}(\mathbf{u}) - \varphi(\mathbf{u}) \quad (6)$$

Having a close-up view of Fig.2, the probability density function of $\Delta\varphi(\mathbf{u})$ can be expressed under the polar coordinate system as follows:

$$p(\Delta\varphi(\mathbf{u})) = \int_0^\infty p_{r,\Delta\varphi(\mathbf{u})}(r, \Delta\varphi(\mathbf{u}))dr = \frac{1}{2\pi\sigma_n^2}\int_0^\infty r \cdot e^{-\frac{1}{2\sigma_n^2}[(r\cos\Delta\varphi(\mathbf{u}) - |F(\mathbf{u})|)^2 + (r\sin\Delta\varphi(\mathbf{u}))^2]} dr \quad (7)$$

where $p(\Delta\varphi(\mathbf{u}))$ is the probability density function of $\Delta\varphi(\mathbf{u})$. (7) shows that $p(\Delta\varphi(\mathbf{u}))$ equals to the sum of the probability density value of the Gaussian noise among $[\varphi(\mathbf{u}), \tilde{\varphi}(\mathbf{u})]$.

As shown in Fig. 2, when the amplitude $|F(\mathbf{u})|$ is far larger compared to σ_n, the above integral become very concentrated. In this case, the outcome approximates to the Gaussian distribution with small variance: most of the values gather in the center. Here we prefer using $ratio = \sigma_n / |F(\mathbf{u})|$ to represent the magnitude relationship between σ_n and $|F(\mathbf{u})|$.



(a) Small ratio (b) Big ratio

Fig. 3. The distribution of $\Delta\varphi(\mathbf{u})$

Fig. 3 shows the distribution of $\Delta\varphi(\mathbf{u})$, $\Delta\varphi(\mathbf{u}) \in [-\pi, \pi]$. When the ratio is small, we can see that the value in the tail of the function is close to zero. This indicates that the probability density value of other period can be ignored. In other words, periods out of the interval $[-\pi, \pi]$ are unlikely to appear. When the ratio becomes bigger, the shape of the function isn't approximately Gaussian because the tail doesn't approach to zero obviously. In this case, periods out of the interval $[-\pi, \pi]$ can't be ignored and it's difficult to fit the curve using a simple non-periodic function. This further proves that the distribution of $\Delta\varphi(\mathbf{u})$ is Gaussian when $|F(\mathbf{u})|$ is far larger than σ_n.

In case of small ratio, $\Delta\varphi(\mathbf{u})$ is correspondingly small most of the time. So we have:

$$\tan \Delta\varphi(\mathbf{u}) \approx |N(\mathbf{u})| / |F(\mathbf{u})| \quad (8)$$

where $N(\mathbf{u})$ is the Fourier transform of the noise. Take the first Taylor expansion term of $\tan \Delta\varphi(\mathbf{u})$:

$$\Delta\varphi(\mathbf{u}) \approx |N(\mathbf{u})| / |F(\mathbf{u})| \quad (9)$$

From the above analysis, the variance $\sigma_{\Delta\varphi(\mathbf{u})}$ is proportional to the *ratio* for specific $|F(\mathbf{u})|$:

$$\sigma_{\Delta\varphi(\mathbf{u})} \approx \sigma_n / |F(\mathbf{u})| \quad (10)$$

Fig. 4 shows the relationship between $\sigma_{\Delta\varphi(\mathbf{u})}$ and *ratio*. We can see that the curve is approximately linear with slope 1.

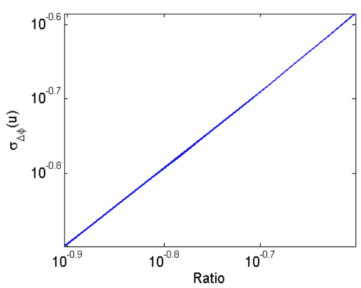


Fig. 4. Relationship between $\sigma_{\Delta\varphi(\mathbf{u})}$ and *ratio*

If $\sigma_n / |F(\mathbf{u})|$ exceeds a threshold ε, the distribution of $\Delta\varphi(\mathbf{u})$ is no longer approximately Gaussian. We should mask out these corresponding phase difference data when estimating the shift parameters because using these data will reduce the precision of the estimation.

In the end, for phase difference data surviving the masking operation by the threshold, the weights can be computed using the following formula:

$$w_i = 1/2\sigma^2_{\Delta\varphi(\mathbf{u}_i)} \approx |F(\mathbf{u}_i)|^2 / 2\sigma_n^2 \quad (11)$$

Here we must point out that $|F(\mathbf{u}_i)|$ is the ground-truth of the spectral amplitude which can't be get straightforwardly. In fact, small error on $|F(\mathbf{u}_i)|$ will not change $\sigma_n / |F(\mathbf{u}_i)|$ largely. So the precision of $|F(\mathbf{u}_i)|$ is not highly required and we only need to predict the rough range of $\sigma_n / |F(\mathbf{u}_i)|$. As we have mentioned earlier, the spectral of the noise-free image pair is strictly identical. So $|F(\mathbf{u}_i)|$ can be roughly computed:

$$|\hat{F}(\mathbf{u}_i)| = mean(|F_1(\mathbf{u}_i)|, |F_2(\mathbf{u}_i)|) \quad (12)$$

We then use the following objective function to compute the shift parameter vector $\Delta\mathbf{x}$:

$$\Delta\mathbf{x} = \arg\min_{\Delta\mathbf{x}} \left\| \mathbf{W}^{1/2} \cdot \Upsilon_\varepsilon((2\pi \cdot \mathbf{U} \cdot \Delta\mathbf{x} - \mathbf{\Psi})) \right\|_2^2 \quad (13)$$

$\mathbf{W}$ is a diagonal matrix with $\mathbf{W}(i,i) = w_i$. Υ_ε is an operator with threshold ε which masks out the contributions from the frequency components that $(\sigma_n / |F(\mathbf{u})|) > \varepsilon$ in the computation, regardless of whether they occur at low or high frequencies.

In summary, a global overview of the proposed registration algorithm can be outlined in Table I.

TABLE I. A SUMMARY OF THE PROPOSED ALGORITHM

Input: A pair of noisy images $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$, σ_n and ε.

Shift Estimation:

(1) Compute the Fourier transforms $F_1(\mathbf{u})$, $F_2(\mathbf{u})$ of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$.

(2) Compute the phase difference between image $f_2(\mathbf{x})$ and the reference image $f_1(\mathbf{x})$ as $\angle(F_2(\mathbf{u}) / F_1(\mathbf{u}))$.

(3) Estimate the ratio of frequency components: $ratio_i = \sigma_n / |\hat{F}(\mathbf{u}_i)|$, where $|\hat{F}(\mathbf{u}_i)| = (|F_1(\mathbf{u}_i)| + |F_2(\mathbf{u}_i)|) / 2$

(4) If $ratio_i \le \varepsilon$, $\angle(F_2(\mathbf{u}) / F_1(\mathbf{u}))$ is accepted in $\mathbf{\Psi}$.

(5) Compute the weights of the elements in $\mathbf{\Psi}$: $w_i = |\hat{F}(\mathbf{u}_i)|^2 / 2\sigma_n^2$

(6) Build and solve the object function:

$$\Delta\mathbf{x} = \arg\min_{\Delta\mathbf{x}} \left\| \mathbf{W}^{1/2} \cdot \Upsilon_\varepsilon((2\pi \cdot \mathbf{U} \cdot \Delta\mathbf{x} - \mathbf{\Psi})) \right\|_2^2$$

Output: Shift parameter $\Delta\mathbf{x}$

IV. EXPERIMENTAL RESULTS

In this section, experimental results on eight conventional

TABLE II. COMPARISON OF THE AVERAGE ABSOLUTE ERROR

σ_n	Hats		Kodim04		Sailboats		Beach		Girl		Parrot		Motor		Airplane	
1	0.013	0.009	0.015	0.008	0.011	0.010	0.017	0.009	0.009	0.006	0.008	0.008	0.007	0.009	0.015	0.008
	0.593	**0.001**	0.607	**0.004**	0.604	**0.009**	0.599	**0.001**	0.606	**0.005**	0.596	**0.003**	0.597	**0.007**	0.597	**0.003**
5	0.083	0.012	0.109	0.010	0.059	0.010	0.106	0.010	0.054	0.007	0.053	0.010	0.047	0.010	0.092	0.008
	0.514	**0.006**	1.046	**0.005**	0.524	**0.003**	0.510	**0.004**	0.551	**0.004**	0.501	**0.004**	0.573	**0.001**	0.599	**0.002**
10	0.153	0.019	0.192	0.017	0.108	0.014	0.195	0.014	0.099	0.012	0.098	0.014	0.086	0.010	0.182	0.011
	0.478	**0.013**	1.227	**0.013**	0.589	**0.009**	0.437	**0.010**	0.492	**0.011**	0.462	**0.011**	0.538	**0.003**	0.642	**0.006**
20	0.301	0.076	0.281	0.056	0.208	0.034	0.383	0.041	0.191	0.042	0.189	0.046	0.166	0.013	0.313	0.026
	0.565	**0.032**	1.813	**0.030**	0.874	**0.027**	0.432	**0.025**	0.449	**0.025**	0.458	**0.030**	0.555	**0.010**	1.353	**0.015**
30	0.462	0.175	0.363	0.132	0.315	0.080	0.572	0.097	0.284	0.095	0.278	0.105	0.250	0.023	0.392	0.056
	0.815	**0.063**	2.263	**0.061**	1.386	**0.051**	0.572	**0.042**	0.485	**0.042**	0.752	**0.050**	0.616	**0.021**	7.177	**0.026**
45	0.702	0.384	0.497	0.297	0.480	0.179	0.843	0.218	0.430	0.212	0.423	0.243	0.392	0.050	0.499	0.129
	14.18	**0.111**	53.27	**0.110**	25.90	**0.086**	0.904	**0.073**	0.725	**0.066**	1.270	**0.087**	4.789	**0.039**	42.11	**0.049**
55	0.844	0.545	0.628	0.432	0.606	0.265	0.983	0.324	0.533	0.308	0.533	0.358	0.473	0.078	0.567	0.194
	79.00	**0.146**	85.28	**0.143**	85.08	**0.109**	24.55	**0.097**	3.272	**0.089**	11.23	**0.117**	3.035	**0.058**	69.21	**0.065**

natural images are presented to evaluate the performance of the proposed method. To make the image circularly symmetric and avoid boundary effects, the images are multiplied by a Tukey window. We then use formula (2) to create the shifted copy of the reference image. The two images are then added zero-mean Gaussian noise with same standard deviation σ_n . The shift parameters are generated by Matlab command **rand** and then multiplied by a magnitude in order to control the maximum displacement. The threshold ε is empirically chosen as 0.2 for the proposed method. For other comparative methods, we choose the empirical parameter settings, which give the best performance for the images.

The proposed method is compared with three representative image registration methods. As shown in Table II, four registration results are reported in each cell. Top left: results of Vandewalle[7]. Top right: results of Keren[2]. Bottom left: results of Lucchese[8]. Bottom right: our proposed algorithm. It is worth emphasizing that Lucchese's method and Keren's method are well-known as noise-robust image registration algorithms. In the experiments, we test six noise levels: σ_n =1, 5, 10, 20, 30, 45, 55. In order to make the experiments more convincing and stable, 100 simulations were performed and then the results are averaged for each σ_n . At last, we compute the average absolute error (pixel) as a criterion to measure the performance of the methods.

As can be seen in Table II, our proposed method outperforms all the four methods evidently in all cases. The performance of Lucchese's method descends rapidly when the noise becomes bigger. However, the precision of our method always stays highly regardless of the noise level. We also test our method on other images. But they are not listed here due to limited space. All the results suggest that our method is much superior to other method in presence of noise.

V. CONCLUSIONS

This paper presents a new noise-robust scheme for image registration. The proposed technique discusses and models how the noise impact the phase of the frequency component. We find out that the phase change is approximately Gaussian distribution under certain circumstances. Based on the analysis, we use a weighted least squares estimator to estimate the shifts between the noisy images. A threshold is set to reject phase data of which the signal-noise ratio is low. Experiments show that the proposed approach can achieve high-precision in face of noise.

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